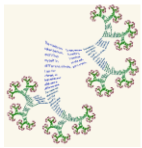
“Historical counterfactuals”
Bristol – August 14, 2012

Infinitesimal turn in a garden of forking paths



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“Historical counterfactuals”
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Corresponding paper (work in progress)

“Real and hyperreal possibility: infinitesimal probabilities in branching time structures”

Slides and paper available at:
http://www.sylviawenmackers.be
/documents/.Bristol/SWenmackersPaper.pdf
/documents/.Bristol/SWenmackersSlides.pdf

Goal

We want to establish a natural link between:
(tense) modal logic &
(non-Archimedean) probability theory

Desired link:
“It is possible that ϕ” is true iff $P(\phi)>0$

We propose definitions for three modal operators (POSS, ⟨r⟩ and ⟨h⟩) and use them to analyze future contingents and historical counterfactuals.

Toy example

Infinite sequence of coin tosses

- Outcome of each toss is contingent; the possibilities are heads (↑) and tails (↓).
- Consider future contingent: “The coin may land heads on each toss.”
- Consider historical counterfactual: Assume first four tosses: ↑, ↑, ↓, and ↑, resp. “If the third toss had been heads, the coin could have landed heads on each toss.”

Work plan

Infinite sequence of coin tosses

1) Apply branching time (BT) structures
2) Apply non-Archimedean probability (NAP)
3) Combine BT and NAP
4) Propose definitions for real/hyperreal possibility that respect the desired link
5) Consider examples of future contingents and historical counterfactuals

& Beyond

6) Evaluate generality of the proposal

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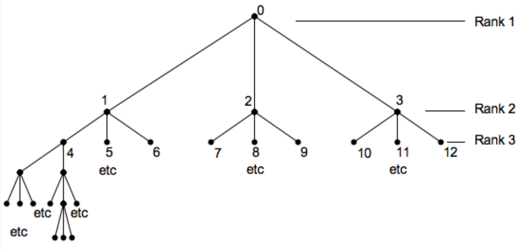
Branching-time (BT) structure

Kripke's suggestion to Prior



Branching-time (BT) structure

On Wed, Aug 8, 2012 at 2:15 AM, Colin McLarty <colin.mclarty@case.edu> wrote:
> Is there any standard preference in reverse mathematics for trees to
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> best, Colin
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> FOM mailing list
> FOM@cs.nyu.edu
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---------- Forwarded message ----------
From: Andrej Bauer <andrej.bauer@andrej.com>
To: Foundations of Mathematics <fom@cs.nyu.edu>
Cc: Colin McLarty <colin.mclarty@case.edu>
Date: Wed, 8 Aug 2012 17:39:03 +0200
Subject: Re: [FOM] Direction of tree growth
Mathematicians seem to grow them upwards, the way Nature does.

Computer scientists grow trees downards, the way they grow in Australia.

This is so because mathematicians can calculate tree height in their heads before drawing the tree, so they know how much blank space to skip. Computer scientists cannot compute, so they just decided to be practical instead.

With kind regards,

Andrej

Branching-time (BT) structure

$\mathcal{F} = \langle M, < \rangle$ (Kripke/Prior)

M: non-empty set of possible moments $\{m_1, m_2, m_3, \ldots\}$

<: earlier-possibly later relation on M that is

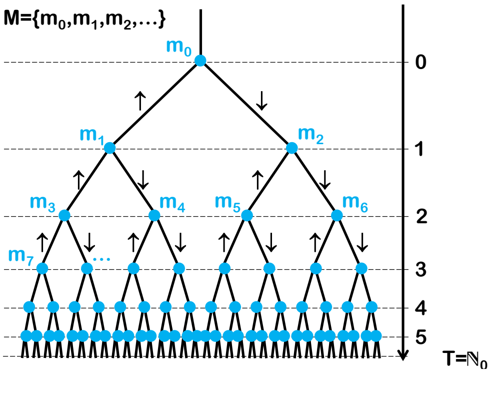
- transitive,
  $\forall m_1, m_2, m_3\ [(m_1 < m_2 \wedge m_2 < m_3) \Rightarrow (m_1 < m_3)]$
- asymmetric,
  $\forall m_1, m_2\ [(m_1 < m_2) \Rightarrow \neg(m_2 < m_1)]$
- backward linear,
  $\forall m, m_1, m_2\ [(m_1 < m \wedge m_2 < m) \Rightarrow (m_1 \leq m_2 \vee m_2 \leq m_1)]$
- historically connected
  $\forall m_1, m_2\ \exists m\ (m \leq m_1 \wedge m \leq m_2)$

(transitive + asymmetric: strict p.o.)

With: $m_1 \leq m_2 \Leftrightarrow (m_1 < m_2 \vee m_1 = m_2)$

BT for tosses with 1 coin

$M = \{m_0, m_1, m_2, \ldots\}$



Branching-time (BT) structure

A history represents a maximal possible course of events – a possible way the world depicted by $\langle M, < \rangle$ could develop.

h is a "history" iff

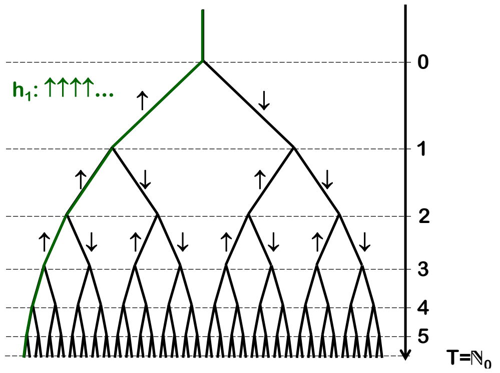
- $h \in \mathcal{P}(M)$
- h is linearly ordered
  $\forall m, m' \in h\ (m < m' \vee m = m' \vee m < m')$
- h is maximal w.r.t. this lin. order

Hence, in a history, any two distinct elements are comparable via <.

Set of all histories: $\mathcal{Hist}$

Müller (2011) "Probabilities in branching structures"

BT for tosses with 1 coin



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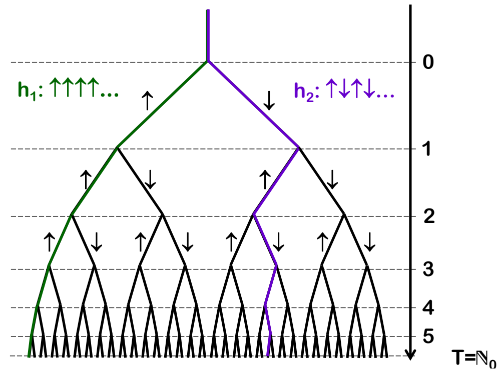

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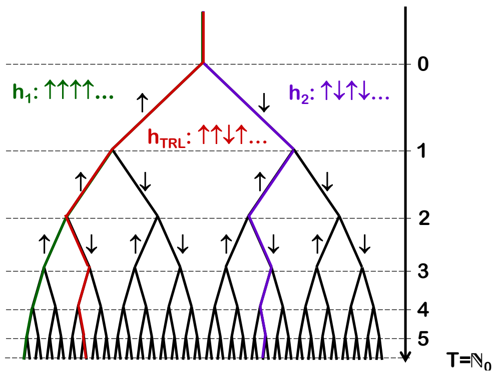


Prior-Thomason semantics for BT structures

Model $\mathcal{M}$ = branching structure ⟨ M, < ⟩ together with valuation V assigning extensions to atomic propositions at the point of evaluation, m/h (*i.e.*, a moment and a history through it; m∈h).

For a stand-alone sentence uttered in a given context (formally: at a moment of context $m_C \in M$), the parameter m is initialized as $m = m_C$.

Müller (2011) "Probabilities in branching structures"

Prior-Thomason semantics for BT structures

Temporal operators

They move the moment of evaluation along the current history of evaluation: change m, keep h.

- **P**: past tense ("it was the case that")
  $\mathcal{M}$, m/h ⊨ Pϕ
  iff ∃m'∈h s.t. m'<m and $\mathcal{M}$, m'/h ⊨ ϕ
- **F**: future tense ("it will be the case that")
  $\mathcal{M}$, m/h ⊨ Fϕ
  iff ∃m'∈h s.t. m<m' and $\mathcal{M}$, m'/h ⊨ ϕ

Müller (2011) "Probabilities in branching structures"

Prior-Thomason semantics for BT structures

Modal operators

They quantify over the possible futures of the moment of evaluation: keep m, change h.

- **Poss**: real possibility,
  $\mathcal{M}$, m/h ⊨ Possϕ
  iff (∃ h' s.t. m∈h') $\mathcal{M}$, m/h' ⊨ ϕ
- **Sett**: necessity or 'settledness'.
  $\mathcal{M}$, m/h ⊨ Settϕ
  iff (∀h' s.t. m∈h') $\mathcal{M}$, m/h' ⊨ ϕ

Observe: Poss = ¬ Sett ¬

Müller (2011) "Probabilities in branching structures"

Division of histories at a moment

Consider a particular moment, m.
Consider the set of histories which contain m, H_m.

If there is a minimal moment, m_0: $\mathcal{H}ist = H_{m_0}$

Müller (2011) "Probabilities in branching structures"

Division of histories at a moment

Consider a particular moment, m.
Consider the set of histories which contain m, H_m.

Define an equivalence relation on H_m, $\equiv_m$:
$\forall h_1,h_2 \in H_m$ $h_1 \equiv_m h_2$ ("are undivided at m")
iff $\exists m' \in h_1 \cap h_2$ s.t. $m<m'$.

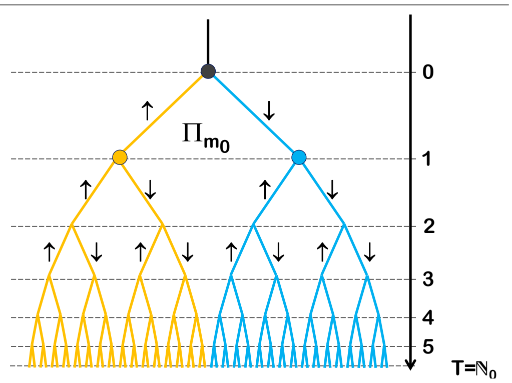
Real possibilities at m are the members of the partition Π_m of H_m induced by $\equiv_m$.

$h_1 \perp_m h_2$ ("split at m")
iff m is maximal in $h_1 \cap h_2$.

m is a choice point
iff Π_m has more than one member
iff there are at least 2 histories splitting at m.

Müller (2011) "Probabilities in branching structures"

BT for tosses with 1 coin



Real possibilities & transitions

Π_m represents the real possibilities open at m; these possibilities form an exhaustive set of mutually exclusive alternatives.

Müller (2011) "Probabilities in branching structures"

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& Beyond

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Standard probability theory

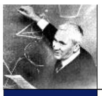
K0 Domain & Range
Probability is a function μ, from a sigma-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ to $[0,1]_{\mathbb{R}}$

K1 Positivity
$\forall A \in \mathcal{P}(\Omega)$, $\mu(A) \geq 0$

K2 Normalization
$A = \Omega \Rightarrow \mu(A) = 1$

K3 Finite Additivity (FA)
$\forall A,B \in \mathcal{P}(\Omega)$, $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$

K4 Continuity (Countable additivity)
$\mu A = \cup_{j \in \mathbb{N}} A_j$ with $(\forall j \in \mathbb{N})$ $A_j \in \mathcal{A}$ and $A_j \subseteq A_{j+1}$
$\Rightarrow \mu(A) = \sup_{j \in \mathbb{N}} \mu(A_j)$



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Kolmogorov space $\langle \Omega, \mathcal{A}, \mu \rangle$

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Kolmogorov space for toy example

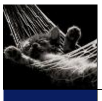
Focus on particular type of events:
cylindrical sets of co-dimension n, defined by

$\forall k \in \{1,\ldots,n\}\ (i_k \in \mathbb{N} \wedge t_k \in \{\uparrow, \downarrow\})$:
$\forall j \in \{1,\ldots,n\},\ k \neq j \Rightarrow i_k \neq i_j$
$C_{(i_1,\ldots,i_n)}^{(t_1,\ldots,t_n)} = \{\omega \in \Omega \mid i_k = t_k\}$

Require that:
$\mu(C_{(i_1,\ldots,i_n)}^{(t_1,\ldots,t_n)}) = 1 / 2^n$

Extend this measure μ to a measure on the sigma-algebra, $\mathcal{A}$, generated by the cylindrical events using Carathéodory's theorem (unique).

We then have the probability space $\langle \Omega, \mathcal{A}, \mu \rangle$.



Non-Archimedean Probability (NAP)

NAP0 Domain & Range
Probability is a function P,
from $\mathcal{P}(\Omega)$ to $[0,1]_{\mathfrak{R}}$ with $\mathfrak{R}$ a superreal field

NAP1 Positivity
$\forall A \in \mathcal{P}(\Omega),\ P(A) \geq 0$

NAP space $\langle \Omega, w, J \rangle$

NAP2 Normalization & Regularity
$\forall A \in \mathcal{P}(\Omega),\ P(A) = 1 \Leftrightarrow A = \Omega$

NAP3 Finite Additivity (FA)
$\forall A,B \in \mathcal{P}(\Omega),\ A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

NAP4 Non-Archimedean Continuity
$\exists$ algebra homomorphism $J: \mathfrak{F}(\mathcal{P}_{fin}(\Omega), \mathbb{R}) \to \mathfrak{R}$
such that $\forall A \in \mathcal{P}(\Omega),\ P(A) = J(\, p(A \mid \cdot)\,)$
$\forall \lambda \in \mathcal{P}_{fin}(\Omega) \setminus \emptyset,\ p(A \mid \lambda) = P(A \cap \lambda) / P(\lambda) \in \mathbb{R}$

NAP space for toy example

0) Problem of interest — Fair coin toss seq.s
1) Choose sample space, Ω $= \{\uparrow, \downarrow\}^{\mathbb{N}}$
2.1) Assign weights to all elements of Ω
$w: \Omega \to \mathbb{R}^+$
$h \mapsto w(h) = 1$
2.2) Assign weight to all *finite* subsets of Ω
$m: \mathcal{P}_{fin}(\Omega) \to \mathbb{R}^+$
$F \mapsto \Sigma_{\omega \in F}\, w(\omega) = \#F$
3) Define probability relative to *finite* non-empty set
$p: \mathcal{P}(\Omega) \times \mathcal{P}_{fin}(\Omega) \setminus \emptyset \to [0,1]$
$(A,F) \mapsto \frac{m(A \cap F)}{m(F)} = \frac{\#(A \cap F)}{\#F}$
4) Absolute probability? Idea: $P(A) = J(\, p(A|F)\,)$

NAP space for toy example

4) J(p(A|F)) is defined as a 'direct limit'

Absolute probability function:

$P: \mathcal{P}(\{H,T\}^{\mathbb{N}}) \to [0,1]_{^*\mathbb{R}}$
$A \mapsto J(\, p(A|\cdot)\,)$

☺ The properties of J can be chosen such that P(A) differs at most by an infinitesimal from μ(A) (where the latter is defined)

V. Benci, L. Horsten, S. Wenmackers
"Non-Archimedean Probability" (Section 5.5)
Accepted in *Milan Journal of Mathematics*
(http://arxiv.org/abs/1106.1524)

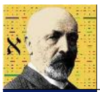
Counting infinite sets

Two principles for comparing sets:

Euclidean part-whole principle
If A is a proper subset of B,
then A is strictly smaller than B.

Humean one-to-one correspondence
If there is a 1-1 correspondence between A and B, then A and B are equal in size.

☹ For infinite sets, these principles are incompatible



Cardinality

Cantor preserved one principle:

~~**Euclidean part-whole principle**~~
~~If A is a proper subset of B,~~
~~then A is strictly smaller than B.~~

Humean one-to-one correspondence
If there is a 1-1 correspondence between A and B, then A and B are equal in size.

This is the basis for 'counting' infinite sets, according to Cantor's cardinality theory

Numerosity

Can we preserve the other principle?

Euclidean part-whole principle
If A is a proper subset of B,
then A is strictly smaller than B.

~~**Humean one-to-one correspondence**~~
~~If there is a 1-1 correspondence between A and B, then A and B are equal in size.~~

The answer is "Yes": this is the basic idea of Benci's numerosity theory.

Numerosity

Basic examples

Assume $\mathbb{N} = \{1,2,3,\ldots\}$ and $num(\mathbb{N}) = \alpha$

Then: $num(\mathbb{N}\backslash\{1\}) = \alpha-1$
$num(\mathbb{Z}) = 2\alpha+1$
$num(\mathbb{N}\times\mathbb{N}) = \alpha^2$
$num(\{1,\ldots,n\}) = n$

Numerosities are defined in such a way that many of the usual algebraic properties of finite real numbers also apply to them.
(They are a special type of hyperreal numbers.)

NAP space for toy example

4) J(p(A|F)) is defined as a 'direct limit'

Absolute probability function:

$P: \mathscr{P}(\{H,T\}^{\mathbb{N}}) \rightarrow [0,1]_{^*\mathbb{R}}$
$A \mapsto J(\, p(A|\cdot)\,)$
$= num(A)/num(2^{\mathbb{N}})$

☺ The properties of J can be chosen such that P(A) differs at most by an infinitesimal from μ(A) (where the latter is defined)

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Infinite sequence of coin tosses

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2) Apply non-Archimedean probability (NAP)
3) **Combine BT and NAP**
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& Beyond

6) Evaluate generality of the proposal

Combining probability & branching structures?

Let *e* be a finite sequence of natural numbers. Think of *e* as a finite data sequence, where data are encoded as natural numbers. Now consider the set $[e]$ of all infinite data streams in $\mathcal{N}$ that extend *e*. Thus, $[e] = \{\varepsilon : \varepsilon | lh(e) = e\}$. I refer to $[e]$ as the *fan* with *handle e* (*Fig. 4.9*).

We may think of the fan with handle *e* as representing the empirical uncertainty of a scientist who has just seen *e*, but whose background assumptions are vacuous (so that $\mathcal{K} = \mathcal{N}$). For all he knows, the actual data sequence may be any infinite extension of *e*. Let *e*, *e′* be finite data sequences. We say $e \subseteq e'$ just in case *e′ extends e* or is identical to *e*. For any two fans, either one includes the other, or the two are disjoint. That is, $[e'] \subseteq [e] \Leftrightarrow e \subseteq e'$.

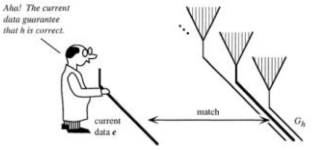


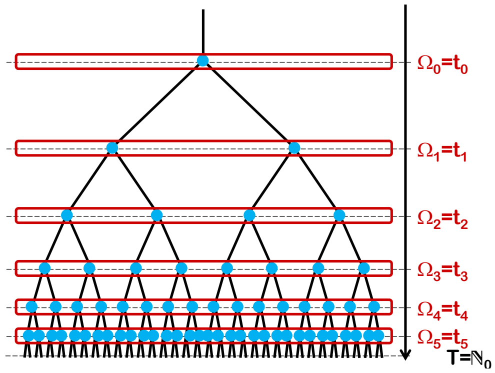
Figure 4.19

K.T. Kelly "The logic of reliable inquiry"

Combining BT with Probability

Idea (1) What is the probability of:
- a particular moment (state of affairs)?
  $P(m)=?$ with $m \in M$ (an infinite set)
- an arbitrary combination of moments
  $P(X)=?$ with $X \in \mathcal{P}(\Omega)$; $\Omega=M$ is an infinite set
- a particular moment m, given a prior moment m'?
  $P(m|m')=?$ with $m'<m \in M$

BT for tosses with 1 coin



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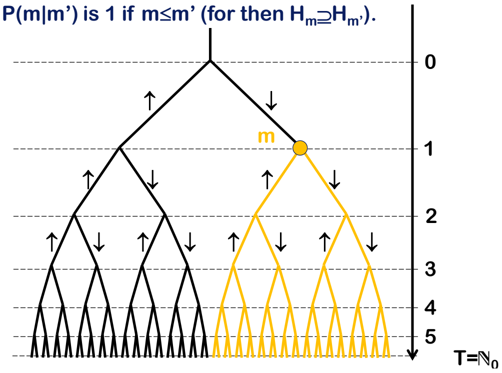
  ~~$P(m)=?$ with $m \in M$ (an infinite set)~~
- an arbitrary combination of moments
  ~~$P(X)=?$ with $X \in \mathcal{P}(\Omega)$; $\Omega=M$ is an infinite set~~

"The probability of a moment, m" can be interpreted as: "the probability of all histories leading to that moment m"
  Recall: $H_m = \{ h \mid h \in \mathcal{Hist} \wedge m \in h\}$;
  $P(m) \equiv P(H_m)=?$ with $m \in M$

Allowing for arbitrary combinations
  $P(X)=?$ with $X \in \mathcal{P}(\Omega)$; $\Omega=\{ H_m \mid m \in M \}$ is infinite

BT for tosses with 1 coin

$P(m|m')$ is 1 if $m \leq m'$ (for then $H_m \supseteq H_{m'}$).



Combining BT with Probability

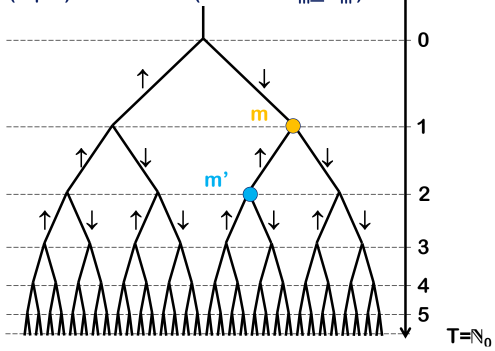
Idea (1 – cont'd)

What is the conditional probability of:
- a particular moment m, given a prior moment m'?
  $P(m|m')=?$ with $m'<m \in M$

$P(m|m')$ can be identified with $P(H_m|H_{m'})$ and computed in a straightforward way from previous probabilities.

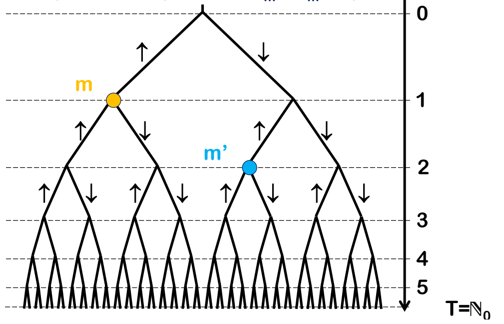
BT for tosses with 1 coin

$P(m|m')$ is 1 if $m \leq m'$ (for then $H_m \supseteq H_{m'}$).



BT for tosses with 1 coin

P(m|m') is 0 if m and m' are incomparable (for then $H_m \cap H_{m'} = \emptyset$).



Combining BT with Probability

Idea (2) What is the probability of:

- a particular history?
  P(h)=? with $h \in \mathcal{Hist}$
- an arbitrary combination of histories?
  P(X)=? with $X \in \mathcal{P}(\Omega)$; $\Omega = \mathcal{Hist}$ is an infinite set

☹ With standard probability theory:
$\forall h \in \mathcal{Hist}$, P({h})=0;
P cannot be defined on all of $\mathcal{P}(\Omega)$.

Combining BT with Probability

Idea (3) What is the probability of:

- a real possibility at moment m
  (*i.e.*, a particular subset of histories)
  P(X)=? with $X \in \Pi_m$ (or equivalently $\in TR_m$)
  $\forall m \in M$, Π_m is a finite set (finite branching)

Müller (2011) investigates the combination of such probability spaces for different moments.

Combining BT with Probability

Müller (2011) p.114:

- Use set TR_m as sample space and employ suitable Boolean σ-algebra and some normalized measure.
- Assumption: BT structure $\langle M, < \rangle$ with <u>finite branching</u>, *i.e.*, $\forall m \in M$ TR_m is finite.
(This does not require M or all H_m to be finite.)

⇒ Probability space: $PR_m = \langle \Omega = TR_m, \mathcal{A} = \mathcal{P}(\Omega), P \rangle$

Goal: Combine two probability spaces defined at different moments. Three cases:

- PR_m and $PR_{m'}$ with m<m' ⎫ Same up to
- PR_m and $PR_{m'}$ with m'<m ⎭ relabeling
- PR_m and $PR_{m'}$ with m and m' incomparable (inconsistent)

Müller (2011) "Probabilities in branching structures"

Combining BT with Probability

Idea (3) What is the probability of:

- a real possibility at moment m?
  (*i.e.*, a particular subset of histories)
  P(X)=? with $X \in \Pi_m$ (or equivalently $\in TR_m$)
  $\forall m \in M$, Π_m is a finite set (finite branching)

Müller (2011) investigates the combination of such probability spaces for different moments.
However, it seems like one could avoid this, by choosing Ω large enough from the start.

$\mathcal{C}(\Omega)$; $\Omega = \mathcal{Hist}$ is an infinite set

Combining BT with Probability

Idea (4) Introduce probability in a way that harmonizes with the modal operators, s.t.:

- **Poss**ϕ corresponds to P(ϕ)>0
- **Sett**ϕ corresponds to P(ϕ)=1
  Recall that these modal operators depend on the context: call this moment m_C

Do we now need a sample space that ranges over ϕ's?
No, $\Omega = \mathcal{Hist}$ can still be used:
"$P(\phi \,@\, m_C)$"
$\equiv P(\{h \in H_{m_C} \mid \exists m' \in h\ (m' > m \wedge \mathcal{M}, m'/h \vDash \phi)\})$

Alternative: quantify over real possibilities

Combining BT with Probability

So, after examining ideas (1) – (4), we can settle on a single sample space:

$$\Omega = \mathcal{Hist}$$

If we want to be able to:

- assign probabilities to all of $\mathcal{P}(\Omega)$,
- make sense of the modal notion of possibility as P>0 [Regularity],

then we cannot use standard probability here, but we can use NAP.

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4) **Propose definitions for real/hyperreal possibility that respect the desired link**
5) Consider examples of future contingents and historical counterfactuals

& Beyond

6) Evaluate generality of the proposal

Possibility: real versus hyperreal

Goal: define modal operators such that

Possϕ is true ⇔ $P(\phi) > 0$
$\Diamond_r\phi$ is true ⇔ $st(P(\phi)) > 0$
$\Diamond_h\phi$ is true ⇔ $P(\phi)>0 \wedge st(P(\phi)) = 0$

Possibility: real versus hyperreal

Proposal for such definitions:

- **Poss**: possibility, (equivalent to previous def.)
  $\mathcal{M}$, m/h ⊨ **Poss**ϕ
  iff $\exists H \subseteq H_m, \forall h' \in H, \mathcal{M}, m/h' \vDash \phi$
- $\Diamond_r$: real possibility, (more general than previous def.)
  $\mathcal{M}$, m/h ⊨ $\Diamond_r\phi$
  iff $\exists$ infinite $H \subseteq H_m, \forall h' \in H, \mathcal{M}, m/h' \vDash \phi$
- $\Diamond_h$: hyperreal possibility, (new)
  $\mathcal{M}$, m/h ⊨ $\Diamond_h\phi$
  iff $\mathcal{M}$, m/h ⊨ (**Poss** $\phi \wedge \neg\Diamond_r\phi$)

Possibility: real versus hyperreal

☺ For the toy example, we obtain indeed:

- **Poss**: possibility,
  $\mathcal{M}$, m/h ⊨ **Poss**ϕ
  iff $P(\phi \mid H_m) > 0$
- $\Diamond_r$: real possibility,
  $\mathcal{M}$, m/h ⊨ $\Diamond_r\phi$
  iff $st(P(\phi \mid H_m)) > 0$
- $\Diamond_h$: hyperreal possibility,
  $\mathcal{M}$, m/h ⊨ $\Diamond_h\phi$
  iff $P(\phi \mid H_m)>0 \wedge st(P(\phi \mid H_m)) = 0$

Work plan

Infinite sequence of coin tosses

1) Apply branching time (BT) structures
2) Apply non-Archimedean probability (NAP)
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& Beyond

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Future contingent

"The coin may land heads on each toss."

Since we do not know the context, we assume this is to be evaluated at moment m_0; hence, we can work with an unconditional probability:

$$P(\{h_{\uparrow\uparrow\uparrow\uparrow\ldots}\}) = 1 / \mathrm{num}(2^{\mathbb{N}})$$

⇒ The relevant probability is a non-zero infinitesimal.

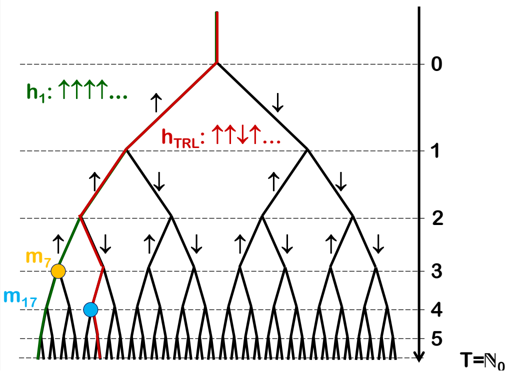
Conclusion: this future contingent is true if the meaning of 'may' corresponds to Poss or ⟨h⟩, but false if the meaning corresponds to ⟨r⟩.

Historical counterfactual

Fact: first tosses are ↑, ↑, ↓, and ↑, resp.

"If the third toss had been heads, the coin could have landed heads on each toss."

BT for tosses with 1 coin



Historical counterfactual

The moment of utterance is m_{17}, but the (false) antecedent moves the moment of evaluation to m_7.
Relevant probability:

$$P(\{h_{\uparrow\uparrow\uparrow\uparrow\ldots}\} \mid H_{m_7}) = 2^3 / \mathrm{num}(2^{\mathbb{N}})$$

⇒ The relevant probability is, again, a non-zero infinitesimal.

Conclusion: this historical counterfactual comes out as true when it is considered as a hyperreal possibility, but it is false when interpreted as a real possibility.

Work plan

Infinite sequence of coin tosses

1) Apply branching time (BT) structures
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& Beyond

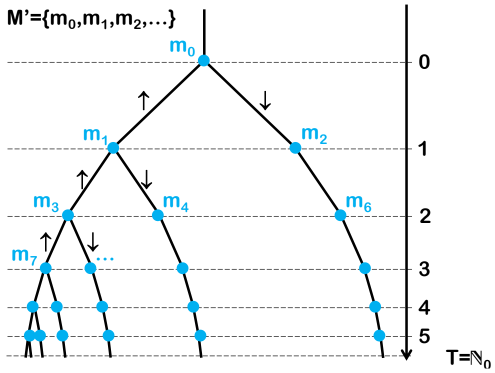
6) Evaluate generality of the proposal

Possibility: real versus hyperreal

Proposed definitions don't work in general!

- Poss: possibility,
  $\mathcal{M}, m/h \vDash \mathrm{Poss}\,\phi$
  iff $\exists H \subseteq H_m, \forall h' \in H, \mathcal{M}, m/h' \vDash \phi$
- ⟨r⟩ : real possibility,
  $\mathcal{M}, m/h \vDash \langle r \rangle \phi$
  iff $\exists$ infinite $H \subseteq H_m, \forall h' \in H, \mathcal{M}, m/h' \vDash \phi$
- ⟨h⟩ : hyperreal possibility,
  $\mathcal{M}, m/h \vDash \langle h \rangle \phi$
  iff $\mathcal{M}, m/h \vDash (\mathrm{Poss}\,\phi \wedge \neg \langle r \rangle \phi)$

Tossing until first tails



Possibility: real versus hyperreal

No 'cardinality gaps' between:
- H and H_m
- set of of choice points in h' and T

- Poss: possibility,
  $\mathcal{M}$, m/h ⊨ Possϕ
  iff ∃H ⊆ H_m, ∀h'∈H, $\mathcal{M}$, m/h' ⊨ ϕ
- ◇r : real possibility,
  $\mathcal{M}$, m/h ⊨ ◇rϕ
  iff ∃ infinite H⊆H_m, ∀h'∈H, $\mathcal{M}$, m/h' ⊨ ϕ
- ◇h : hyperreal possibility,
  $\mathcal{M}$, m/h ⊨ ◇hϕ
  iff $\mathcal{M}$, m/h ⊨ (Poss ϕ ∧ ¬◇rϕ)

Realistic examples?

Infinite sequence of coin tosses:
  highly idealized concept (toy example!)
  (Infinite *Hist*)

Relation to real life = ?
We may encounter very long but finite sequences of contingent events
Model: sequence of, *e.g.*, 10 000 coin tosses
  (Finite *Hist*)

With relative analysis, same qualitative picture: distinguish between possible but highly unlikely events and possible events that have a more substantive probability.

Thank you for your attention

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Slides and paper available at:
http://www.sylviawenmackers.be
/documents/.Bristol/SWenmackersPaper.pdf
/documents/.Bristol/SWenmackersSlides.pdf

Construction of a model

0) Problem of interest — Fair coin toss seq.s

1) Choose sample space, Ω = $\{\uparrow,\downarrow\}^{\mathbb{N}}$

2.1) Assign weights to all elements of Ω
  w: Ω → $\mathbb{R}^+$
  h ↦ w(h) = 1

2.2) Assign weight to all *finite* subsets of Ω
  m: $\mathcal{P}_{fin}(\Omega) \to \mathbb{R}^+$
  F ↦ $\Sigma_{\omega\in F}$ w(ω) = #F

3) Define probability relative to *finite* non-empty set
  p: $\mathcal{P}(\Omega) \times \mathcal{P}_{fin}(\Omega)\backslash\varnothing \to [0,1]$
  (A,F) ↦ $\frac{m(A\cap F)}{m(F)} = \frac{\#(A\cap F)}{\#F}$

4) Absolute probability? Idea: P(A) = J(p(A|F))

Construction of a model

4) J(p(A|F)) is defined as a 'direct limit'

4.1) Choose directed set $\langle \Lambda, \subseteq \rangle$ ($\subseteq$ is preorder on Λ)

In addition, require:

$\Lambda \subseteq \mathcal{P}_{fin}(\Omega) \backslash \varnothing$

$\forall F \in \mathcal{P}_{fin}(\Omega), \exists \lambda \in \Lambda: F \subset \lambda$

$\Lambda = \{ \lambda_{n,F} \mid n \in \mathbb{N} \wedge F \in \mathcal{P}_{fin}(\Omega) \backslash \varnothing \}$

where $\lambda_{n,F} = \{ \alpha \oplus \beta \mid \alpha \in \{\uparrow,\downarrow\}^n \wedge \beta \in F \}$.

Observe: A fixed $\Rightarrow$ p(A| .) is a 1-place function

Consider φ = p(A| .): $\Lambda \rightarrow [0,1]$

4.2) Set of *all* functions '$\varphi: \Lambda \rightarrow \mathbb{R}$' = $\mathcal{F}(\Lambda,\mathbb{R})$

$\Rightarrow \langle \mathcal{F}(\Lambda,\mathbb{R}), +, \times, \leq \rangle$ is a partially ordered ring

Construction of a model

4) J(p(A|F)) is defined as a 'direct limit'

4.3) Construct a field for the range of P:

4.3.1) Define ideal of $\mathcal{F}(\Lambda,\mathbb{R})$ (negligible elements)

$\mathfrak{I}_0 = \{ \varphi \in \mathcal{F}(\Lambda,\mathbb{R}) \mid \exists \lambda_0 \in \Lambda, \forall \lambda \supseteq \lambda_0 : \varphi(\lambda)=0 \}$

4.3.2) Extend the ideal to a maximal ideal, $\mathfrak{I}_{max}$

Exists, but not unique (via Zorn's lemma)

4.3.3) Define equivalence classes on $\mathcal{F}(\Lambda,\mathbb{R})$

$[\varphi]_{\Omega,\Lambda} = \{ \psi \in \mathcal{F}(\Lambda,\mathbb{R}) \mid \exists \varepsilon \in \mathfrak{I}_{max} : \varphi = \psi + \varepsilon \}$

4.3.4) Define $\mathfrak{R}_{\Omega,\Lambda} = \mathcal{F}(\Lambda,\mathbb{R}) / \mathfrak{I}_{max}$

$= \{ [\varphi]_{\Omega,\Lambda} \mid \varphi \in \mathcal{F}(\Lambda,\mathbb{R}) \} = {}^*\mathbb{R} \supseteq \mathbb{R}$

This set is an ordered field

When Ω is infinite, it is non-Archimedean

Construction of a model

4) J(p(A|F)) is defined as a 'direct limit'

4.4) Define: $J(\varphi(\cdot)) = \lim_{\lambda \uparrow \Omega} \varphi(\lambda) = [\varphi]_{\Omega,\Lambda} \in \mathfrak{R}_{\Omega,\Lambda}$

In particular:

$J: \mathcal{F}(\Lambda,\mathbb{R}) \rightarrow \mathfrak{R}_{\Omega,\Lambda}$

$p(A|\cdot) \mapsto \lim_{\lambda \uparrow \Omega} p(A|\lambda) = [p(A|\cdot)]_{\Omega,\Lambda} \in \mathfrak{R}_{\Omega,\Lambda}$

4.5) Define absolute probability function:

$P: \mathcal{P}(\Omega) \rightarrow \mathfrak{R}_{\Omega,\Lambda}$

$A \mapsto J(p(A|\cdot)) = [p(A|\cdot)]_{\Omega,\Lambda}$

$P: \mathcal{P}(\{H,T\}^{\mathbb{N}}) \rightarrow [0,1]_{^*\mathbb{R}}$

$A \mapsto J(p(A|\cdot))$

$= num(A)/num(2^{\mathbb{N}})$

☺ Due to our choice of Λ, P(A) differs at most by an infinitesimal from μ(A) (where the latter is defined)